effects are important and should be taken into account in designing parallel plate systems in accordance with the results of this paper.

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NOTATION

- b = half the distance between the plates
- Br_n = Brinkman number for a power-law fluid
- C_p = specific heat
- h = film heat transfer coefficient
- k = thermal conductivity
- m = consistency index
- n = power-law index
- Nu = Nusselt number
- P = pressure
- Pe = Peclet number
- T = temperature
- T_0 = temperature at $\psi = 0$
- T_w = temperature at the wall
- U_x = velocity in the x-direction
- U_{max} = maximum velocity
- x = coordinate in the direction of the flow
- y = coordinate perpendicular to the direction of the flow
- β = viscous dissipation parameter, defined by Equation (11)
- $\Delta\eta$ = step size in the η-direction
- $\Delta\psi$ = step size in the ψ-direction
- η = dimensionless variable defined by Equation (6)
- θ = dimensionless temperature defined by Equation (7)
- θ_b = bulk temperature, defined by Equation (18)
- ρ = density
- τ = shear stress
- ψ = dimensionless distance, defined by Equation (8)

LITERATURE CITED

Bird, R. B., "Viscous Heat Effects in Extrusion of Molten Plastics," *Soc. Plastics Eng. J.*, **11**, 35 (1955).

Forsyth, T. H., and N. F. Murphy, "Temperature Profiles of Molten Flowing Polymers in a Heat Exchanger," *AIChE J.*, **15**, 758 (1969).

Gavis, J., and R. L. Laurence, "Viscous Heating of a Power-Law Liquid in Plane Flow," *Ind. Eng. Chem. Fundamentals*, **7**, 525 (1969).

Kim, H. T., and E. A. Collins, "Temperature Profiles for Polymer Melts in Tube Flow. Part II. Conduction and Shear Heating Corrections," *Poly. Eng. Sci.*, **11**, 83 (1971).

Prins, J. A., J. Mulder, and J. Schenk, "Heat Transfer in Laminar Flow Between Parallel Plates," *Appl. Sci. Res.*, **A2**, 431 (1950).

Smorodinskii, E. L., and G. B. Froishteter, "Analytical Solution of the Problem of the Laminar Heat Transfer of Nonlinearly Viscoplastic Fluids Taking Account of the Dissipation of Kinetic Energy," *Theoret. Osnovy Khimicheskoi Tekn.*, **5**, 542 (1971).

Suckow, W. H., P. Hrycak, and R. G. Griskey, "Heat Transfer to Polymer Solutions and Melts Flowing Between Parallel Plates," *Poly. Eng. Sci.*, **11**, 401 (1971).

Tien, C., "The Extension of Couette Flow Solution to Non-Newtonian Fluid," *Can. J. Chem. Eng.*, **39**, 45 (1961).

———, "Laminar Heat Transfer of Power-Law Non-Newtonian Fluid—The Extension of Graetz-Nusselt Problem," *ibid.*, **40**, 130 (1962).

Toor, H. L., "Heat Generation and Conduction in the Flow of a Viscous Compressible Liquid," *Trans. Soc. Rheol.*, **1**, 177 (1957).

———, "Heat Transfer in Forced Convection with Internal Heat Generation," *AIChE J.*, **4**, 319 (1958).

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Comparison of Various Ways of Model Building of a Regenerator

J. BRASZ and A. U. KHAN

Department of Applied Physics
Eindhoven University of Technology
P.O. Box 513, Eindhoven, Netherlands

For regenerators the dimensionless heat balances of the solid and gas, under usual assumptions, result in the following set of linear partial differential equations:

$$\frac{\partial S(z,t)}{\partial t} = G(z,t) - S(z,t) \quad (1)$$

$$\frac{\partial G(z,t)}{\partial z} = S(z,t) - G(z,t) \quad (2)$$

When the entrance gas temperature is constant and equal to the zero point of our temperature scale and when the initial solid temperature has the same value for all z, the normalized boundary conditions can be written as

$$S(z,t)|_{t=0} = 1 \quad (3)$$

$$G(z,t)|_{z=0} = 0 \quad (4)$$

The basic equations underlying many calculations in the simulation of industrial gas-solid cross flow heat exchangers are similar to the regenerator equations. In dynamic studies of these heat exchangers, for example, in the study of Rademaker et al. (1970) about a clinker cooler and the study of Voskamp and Brasz (1971) concerning a pellet indurating plant, Equations (1) to (4) have to be solved many times. Therefore, it is desirable to look for the most efficient solution method of these equations. Three methods of solving Equations (1) to (4), namely, an analytical solution, a finite difference solution, and a numerical inversion solution are compared on accuracy and computer time.

THE ANALYTICAL SOLUTION

Many analytical solutions have been given to the set of Equations (1) to (4). All of them are in a series form, so that for a definitive answer a truncation must be made. In our opinion the most efficient analytical solution is obtained by Kohlmayr (1968) using double Laplace transforms. There results the following expression for the gas temperature response function:

$$G(z,t) = \lim_{n\to\infty}\left\{1 - e^{-z-t}\sum_{i=0}^{n}\frac{t^i}{i!}\sum_{j=0}^{i}\frac{z^j}{j!}\right\} \qquad (5)$$

THE FINITE DIFFERENCE METHOD

According to the finite difference method the height z of the regenerator is divided into n layers. For the ith layer the mean normalized solid temperature is written as $S_i(z,t)$ and the incoming and outgoing normalized gas temperatures are denoted by $G_{i-1}(z,t)$ and $G_i(z,t)$, respectively. Instead of the two original partial differential Equations (1) and (2) with boundary conditions (3) and (4), the following n differential and n algebraic equations result:

$$\frac{dS_i(z,t)}{dt} = \frac{G_{i-1}(z,t) + G_i(z,t)}{2} - S_i(z,t); \qquad i = 1, 2, \ldots, n$$

$$\frac{G_i(z,t) - G_{i-1}(z,t)}{z/n} = S_i(z,t) - \frac{G_{i-1}(z,t) + G_i(z,t)}{2}; \qquad i = 1, 2, \ldots, n$$

with boundary conditions

$$S_i(z,t)|_{t=0} = 1; \qquad i = 1, 2, \ldots, n$$

$$G_i(z,t)|_{i=0} = 0$$

By Laplace transformation with respect to time and using the initial conditions, the following equations can be obtained:

$$\overline{G}_0(z,q) = 0$$

$$\overline{S}_i(z,q) = \frac{\tau_+}{\tau_+ q + 1} + \frac{\overline{G}_{i-1}(z,q)}{\tau_+ q + 1} \qquad i = 1, 2, \ldots, n$$

$$\overline{G}_i(z,q) = \frac{\tau_-}{\tau_+}\overline{G}_{i-1}(z,q) + \frac{\tau_+ - \tau_-}{\tau_+}\overline{S}_i(z,q) \qquad i = 1, 2, \ldots, n$$

where

$$\tau_+ = 1 + \frac{z}{2n}$$

$$\tau_- = 1 - \frac{z}{2n}$$

Rearranging these equations, the Laplace transform of the outcoming gas temperature can be written as

$$\begin{aligned}\overline{G}_n(z,q) &= \frac{1}{q} - \frac{1}{q}\left(\frac{\tau_- q + 1}{\tau_+ q + 1}\right)^n \\ &= \frac{1}{q} - \frac{1}{q}\left\{\frac{\tau_-}{\tau_+}\left(1 + \frac{1/\tau_- - 1/\tau_+}{q + 1/\tau_+}\right)\right\}^n \\ &= \frac{1}{q} - \frac{1}{q}\left(\frac{\tau_-}{\tau_+}\right)^n \left\{1 + \sum_{r=1}^{n}\binom{n}{r}\left(\frac{1/\tau_- - 1/\tau_+}{q + 1/\tau_+}\right)^r\right\}\end{aligned}$$

Using the Laplace transform pair

$$L^{-1}\left\{\frac{1}{q}\,\frac{1}{(q + 1/\tau_+)^r}\right\} = \tau_+^{\,r}\left\{1 - \sum_{m=0}^{r-1}\frac{t^m\, e^{-t/\tau_+}}{\tau_+^{\,m}\, m!}\right\}$$

and the algebraic identity

$$1 + \sum_{r=1}^{n}\binom{n}{r}\left\{\frac{\tau_+}{\tau_-} - 1\right\}^r = \left(1 + \frac{\tau_+ - \tau_-}{\tau_-}\right)^n = \left(\frac{\tau_+}{\tau_-}\right)^n$$

the following representation of the gas temperature response function is found:

$$G_n(z,t) = e^{-t/\tau_+}\left(\frac{\tau_-}{\tau_+}\right)^n \sum_{r=1}^{n}\binom{n}{r}\left(\frac{\tau_+ - \tau_-}{\tau_-}\right)^r \sum_{m=0}^{r-1}\frac{t^m}{\tau_+^{\,m}\, m!} \qquad (6)$$

THE NUMERICAL INVERSION SOLUTION

After Laplace transformation with respect to z Equations (1) to (4) give

$$\frac{d\overline{S}(p,t)}{dt} = \overline{G}(p,t) - \overline{S}(p,t)$$

$$p\overline{G}(p,t) = \overline{S}(p,t) - \overline{G}(p,t)$$

$$\overline{S}(p,t)|_{t=0} = \frac{1}{p}$$

Eliminating $\overline{S}(p,t)$ and solving the differential equation in t gives the following expression for $\overline{G}(p,t)$:

$$\overline{G}(p,t) = \frac{1}{p(p+1)}\, e^{-\frac{p}{p+1}t}$$

Using the definition of $\overline{G}(p,t)$ as an integral, we get

$$\int_0^\infty G(z,t)\, e^{-pz}\, dz = \frac{1}{p(p+1)}\, e^{-\frac{p}{p+1}t}$$

which becomes after substitution $x = e^{-z}$

$$\int_0^\infty x^{p-1}\, G(-\ln(x), t)\, dx = \frac{1}{p(p+1)}\, e^{-\frac{p}{p+1}t}$$

The unknown in this equation is $G(-\ln(x), t)$. This integral equation can be solved numerically using a finite sum of terms existing of the value of the integrand in a number of points x_i times a weighting factor w_i. The x_i and w_i have to be chosen so that the approximation

$$\sum_{i=1}^{n} w_i x_i^{p-1} G(-\ln(x_i), t) \simeq \frac{1}{p(p+1)} e^{-\frac{p}{p+1}t}$$

is as accurate as possible. According to Bellman et al. (1966) the roots of the shifted Legendre polynomials of degree n are taken for x_i with appropriate weighting factors w_i. Let p now take n different values, namely, $p = 1, 2, \ldots, n$, then the last equation is a set of n equations with n unknowns $G(z_i, t)$, $z_i = -\ln x_i$. After a matrix inversion we obtain:

$$G(z_i, t) = \sum_{p=1}^{n} A_{ip} \frac{1}{p(p+1)} e^{-\frac{p}{p+1}t} \qquad (7)$$

The matrix A_{ip} has to be calculated only once from the x_i and w_i. The from the choice of x_i and w_i resulting matrices A_{ip} are given in Appendix V of Bellman et al. (1966) for $n = 3, 4, \ldots, 15$.

RESULTS AND CONCLUSIONS

The accuracy of G_{appr}, the approximate solutions of Equations (5), (6), and (7) in the points (z_i, t_j) where $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, l$ can be expressed quantitatively by a quadratic error norm I_q:

$$I_q = \left[\frac{1}{k \cdot l} \sum_{i=1}^{k} \sum_{j=1}^{l} \{G_{exact}(z_i, t_j) - G_{appr}(z_i, t_j)\}^2\right]^{1/2}$$

The exact solution $G_{exact}(z_i, t_j)$ is obtained by calculating Equations (5), (6), and/or (7) for such a large value of n that the series has reached its limit. In Figure 1 the quadratic error norm I_q with $k = l = 10$ and $z_i = 2, 4, \ldots, 20$ and $t_j = 2, 4, \ldots, 20$ is plotted for different values of n in Equations (5), (6), and (7) against the needed computer time, expressed in machine independent cycletimes. The various values of n which are used in the calculations are indicated in the plot.

According to Figure 1 the finite difference method needs less computer time than the other methods when the required accuracy norm I_q is greater than 4×10^{-4}. When $I_q < 4 \times 10^{-4}$ the analytical solution is the fastest solution method. For practical calculations where an accuracy of 1% is good enough, the finite difference solution is most satisfactory.

For an accuracy of $I_q = 6 \times 10^{-4}$ the numerical inversion method needs 15 terms and the finite difference method 25 terms. In building an analogue model of Equations (6) and (7), the number of needed integrators is of first importance. This number equals the number n of needed terms from (6) and (7). Therefore, for analogue simulation studies solutions using numerical inversion techniques may be more efficient than the usual finite difference solutions. This result corresponds with the experience of Aurora (1971), who found in modeling a parallel flow fluid-fluid heat exchanger a higher accuracy with an analogue model using 15 integrators according to the numerical inversion technique than with an analogue model using 24 integrators according to the finite difference approach.

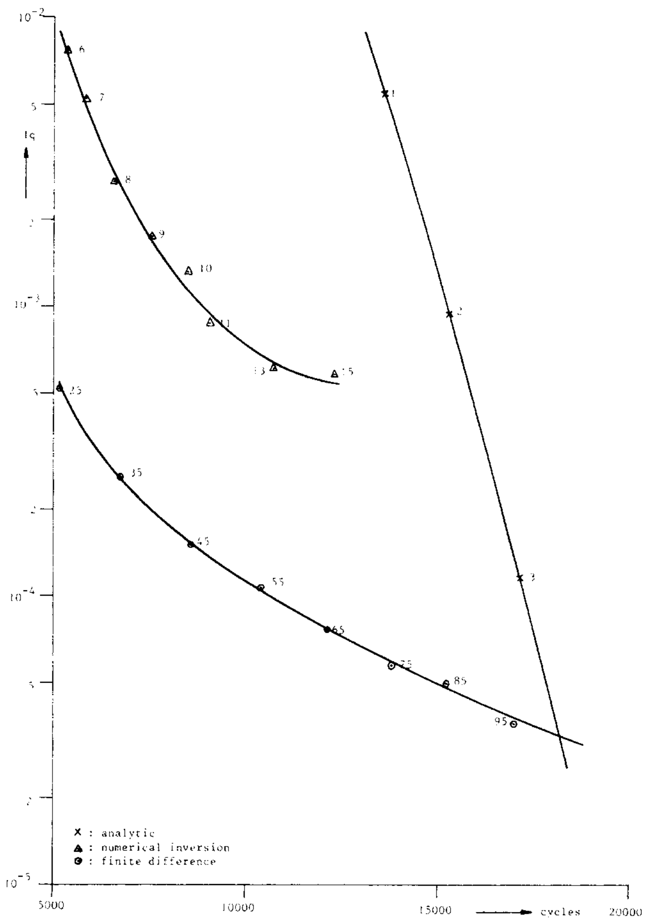

Fig. 1. Accuracy of the 3 solution methods according to the quadratic norm I_q versus needed cycles on the digital computer.

NOTATION

$G(z, t)$ = normalized gas temperature
I_q = quadratic error norm
p = Laplace variable of z
q = Laplace variable of t
$S(z, t)$ = normalized solid temperature
t = dimensionless time coordinate
w_i = weighting factor
x = e^{-z}
z = dimensionless place coordinate
τ_+ = abbreviation
τ_- = abbreviation

LITERATURE CITED

Aurora, A. K., "The simulation of a distributed parameter system on an analogue computer," Philips Intern. Inst. of Tech. Studies, report 394 (1971).

Bellman, R., R. E. Kabala, J. A. Lockett, *Numerical Inversion of the Laplace Transform*, Elsevier, New York (1966).

Kohlmayr, G. F., "Properties of Transient Heat Transfer (Single Blow) Temperature Response Function," *AIChE J.*, **14**, 499 (1968).

Rademaker, O., L. H. Goessens, J. H. Voskamp, and A. C. P. Debie, "Dynamics and Control of a Clinker Cooler," *Automatica*, **6**, 231 (1970).

Voskamp, J. H., and J. Brasz, "Control Analysis of a Pellet Indurating Machine," Internal THE report Eindhoven Univ. Techn. (October, 1971) to be published in *Iron Steel*.

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